Snyder noncommutative space-time from two-time physics

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We show that the two-time physics model leads to a mechanical system with Dirac brackets consistent with the Snyder noncommutative space. An Euclidean version of this space is also obtained and it is shown that both spaces have a dual system describing a particle in a curved space.

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I. INTRODUCTION

Inspired by a conformal field theory, R. Marnelius [1] built a classical mechanics model having the conformal group as the global symmetry and the symplectic group $S_P(2)$ as the local one. This model has interesting unusual properties. One of them is that it must have two time coordinates; that is why it is normally called the two-time physics (2T) model. By imposing different gauge conditions on this, one can obtain systems such as the relativistic particle with mass and the massless free particle in the AdS space-time. In this sense the 2T model can be used as a toy model for unification. Supersymmetric extensions of the 2T model can be found in Ref. [2]. Recently, I. Bars and co-workers reinvented the 2T model in string theory [3] and carried out several extensions in different contexts (see Refs. [4, 5] and references therein). Ref. [6] also deals with the same problem. In another work, M. Montesinos, C. Rovelli and T. Thiemann proposed a classical mechanics model simulating the gauge structure of general relativity. In this, the gauge group is $SL(2,R)$ [7] and, since $SL(2,R)$ is isomorphic to $S_P(2)$, this model is analogous to the 2T. As it is, the 2T model has several interesting properties one would like to see in a realistic model.

From different results in string theory [8], the possibility that the space-time at short distances is noncommutative has been extensively studied recently. R. Snyder [9] investigated these ideas first and built a noncommutative Lorentz invariant discrete space-time: the so called Snyder space. Contrarily to the noncommutative spaces from string theory, in Snyder space the noncommutativity depends on the space-time. After the work of Kontsevich [10], Snyder-like spaces in the sense of noncommutativity have attracted great attention. Snyder space is also interesting because it can be mapped to the k-Minkowski space-time [11]. This space-time is a realization of the "Doubly Special Relativity" theory, which is a new proposal to deal with quantum gravity phenomena [12]. An important result from loop quantum gravity, in addition, is that it leads to discrete geometric quantities [13], and in this sense the discreteness of Snyder space becomes also attractive.

We show in this investigation how, by imposing an alternative gauge condition on the 2T model, one gets to a mechanical system with Dirac brackets consistent with the commutation rules of the Snyder noncommutative space. Using other gauge conditions, we also show that an Euclidean version of the Snyder space can be obtained. Then, by exploiting the symmetries of the Hamiltonian, we conclude that each system has a dual. For the Snyder space the dual system is the massless particle in the AdS space, but for the Euclidean Snyder it is the non-linear sigma model in one dimension.

The work in this paper is organized as follows. In Section 2 a brief introduction to the 2T model is provided. Then, in Section 3 the gauge conditions to get to the Snyder space are given. The analogous conditions, but to obtain the Euclidean Snyder space are determined in Section 4. In Section 5 we show that both of these spaces have a dual system; and finally in Section 6 we summarize our results.

II. THE 2T MODEL

Let us first review some properties of the 2T action and its symmetries.

A. 2T action

The action for the 2T model is defined as the Hamiltonian action

$$S=\int_{\tau_1}^{\tau_2} d\tau\left[\dot{X}\cdot P-\left(\lambda^1\frac{1}{2}P^2+\lambda^2 X\cdot P+\lambda^3\frac{1}{2}X^2\right)\right], \tag{1}$$

with the Hamiltonian given by

$$H_{2T}=\left(\lambda^1\frac{1}{2}P^2+\lambda^2 X\cdot P+\lambda^3\frac{1}{2}X^2\right), \tag{2}$$

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where $\lambda^1, \lambda^2, \lambda^3$ are Lagrange multipliers. From this, one can obtain the equations of motion

$$\dot{X}^M = \lambda^1 P^M + \lambda^2 X^M \,, \tag{3}$$
$$\dot{P}^M = -\lambda^2 P^M - \lambda^3 X^M \,, \tag{4}$$
$$P^2 \approx X^2 \approx X \cdot P \approx 0 \,, \tag{5}$$

where the symbol of weak equivalence ($\approx$) has been used in the sense of Dirac [14, 15]. Now, by defining

$$\phi_1 = \frac{1}{2} P^2 \,, \quad \phi_2 = X \cdot P \,, \quad \phi_3 = \frac{1}{2} X^2 \,, \tag{6}$$

and considering that the Poisson brackets are given by: $\{X_M, P_N\} = \eta_{MN}$, and zero otherwise, with η_{MN} being a flat metric, it can be seen that the algebra

$$\{\phi_2, \phi_3\} = -2\phi_3 \,, \quad \{\phi_2, \phi_1\} = 2\phi_1 \,, \quad \{\phi_1, \phi_3\} = -\phi_2 \,, \tag{7}$$

holds. That is, all three constraints are first class. Eq. (7) represents the Lie algebra of the $S_P(2)$ group which is formed by the 2×2 matrices with determinant one. If one redefines variables as

$$H_1 = \phi_1 \,, \quad H_2 = -\phi_3 \,, \quad D = \phi_2 \,, \tag{8}$$

the Lie algebra of the $SL(R, 2)$ is obtained. This has been already proposed as a toy model simulating the gauge group of general relativity [7].

Now, if we consider the Euclidean or Minkowski metrics as the background space, the surface defined by Eq. (5) is trivial. Therefore, the simplest metric giving a non-trivial surface is the flat metric with two time coordinates. Throughout this work we will assume this metric only. If the configuration space has dimensionality $D = d + 2$, a flat metric η_{MN} with signature

$$\mathrm{sig}(\eta) = (-, -, +, \cdots, +) \,, \tag{9}$$

must be used. The coordinates of the phase space can be taken as

$$X^M = (X^{0'}, X^{1'}, X^0, X^i) \,,$$
$$P^M = (P^{0'}, P^{1'}, P^0, P^i) \,, \quad (i = 1, \ldots, d-1) \,, \tag{10}$$

where the zeroes are associated with the time coordinates.

In principle the phase space of the system has $2(d+2)$ independent coordinates. However, as there are three first-class constraints, six degrees of freedom must be subtracted. Therefore, there are $2(d-1)$ effective degrees of freedom and so the configuration space has $(d-1)$ independent coordinates.

B. Symmetries

The equations of motion (3) and (4) can be rewritten as

$$\frac{d}{dt} \begin{pmatrix} X^M \\ P^M \end{pmatrix} = A(t) \begin{pmatrix} X^M \\ P^M \end{pmatrix} \,, \tag{11}$$

with $A(t) = \begin{pmatrix} \lambda^2 & \lambda^1 \\ -\lambda^3 & -\lambda^2 \end{pmatrix}$. By performing a gauge transformation with an arbitrary matrix of $Sp(2)$,

$$\begin{pmatrix} \bar{X}^M \\ \bar{P}^M \end{pmatrix} = U(t) \begin{pmatrix} X^M \\ P^M \end{pmatrix} \,, \tag{12}$$

where $U(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad ad - bc = 1$, one gets to the transformed equations of motion

$$\frac{d}{dt} \begin{pmatrix} \bar{X}^M \\ \bar{P}^M \end{pmatrix} = \bar{A}(t) \begin{pmatrix} \bar{X}^M \\ \bar{P}^M \end{pmatrix} \,, \tag{13}$$

where

$$\bar{A}(t) = U(t) A(t) U(t)^{-1} - U(t) \frac{dU(t)^{-1}}{dt} \,. \tag{14}$$

It can be easily seen that $A(t)$ transforms as a connection under the gauge transformation $U(t)$ and that the equations of motion (11) are invariant under this gauge transformation as well.

Now, the action in Eq. (1), when rewritten in terms of the transformed variables, takes the form

$$\begin{aligned} S &= \int_{\tau_1}^{\tau_2} d\tau \left[\dot{X} \cdot P - \left(\lambda^1 \frac{1}{2} P^2 + \lambda^2 X \cdot P + \lambda^3 \frac{1}{2} X^2 \right) \right] \\ &= \int_{\tau_1}^{\tau_2} d\tau \left[\frac{d\bar{X}}{d\tau} \cdot \bar{P} - \left(\bar{\lambda}^1 \frac{1}{2} \bar{P}^2 + \bar{\lambda}^2 \bar{X} \cdot \bar{P} + \bar{\lambda}^3 \frac{1}{2} \bar{X}^2 \right) \right. \\ &\quad \left. + \frac{d}{d\tau} \left(-ab \frac{1}{2} \bar{P}^2 + bc \bar{X} \cdot \bar{P} - dc \frac{1}{2} \bar{X}^2 \right) \right] \,, \end{aligned} \tag{15}$$

where $\bar{\lambda}^i$ is given by Eq. (14). Thus, up to a boundary term, the action in Eq. (1) is invariant under the gauge transformations (12) and (14). On the other hand, the quantities $X \cdot P$, X^2, P^2 and $\dot{X} \cdot P$ are clearly invariant under global transformations Λ that satisfy

$$\Lambda^T \eta \Lambda = \eta \,, \tag{16}$$

with the signature η defined in Eq. (9). Thus, the action in Eq. (1) is invariant under global transformations of $SO(2, d)$. It can be shown that in phase space the generators of this symmetry are

$$L^{MN} = X^M P^N - X^N P^M \,, \tag{17}$$

which satisfy the conformal algebra [16] and are conserved quantities. Moreover, they satisfy $\{L^{MN}, \phi_i\} = 0$, i.e. they are gauge invariant.

III. SNYDER SPACE

Let us now consider the gauge conditions to get the Snyder space

$$P_{1'} = L = \mathrm{const.} \,, \qquad X_{1'} = 0 \,. \tag{18}$$

Substituting them into the equations of motion (3) and (4) we obtain

$$\lambda^2 = \lambda^1 = 0\,. \tag{19}$$

By using Eq. (5) it can be seen that the independent reduced equations of motion are

$$\dot{X}^\mu = 0\,, \tag{20}$$
$$\dot{P}_\mu = -\lambda^3 X_\mu\,, \tag{21}$$
$$\phi_3 = \frac{1}{2}G_{\mu\nu}X^\mu X^\nu \approx 0, \quad G_{\mu\nu} = \left(\eta_{\mu\nu} - \frac{P_\mu P_\nu}{P_\alpha P^\alpha + L^2}\right) \tag{22}$$

with the dependent variables being

$$P_{0'} = \sqrt{P_\mu P^\mu + L^2}\,, \quad X_{0'} = \frac{P_\mu X^\mu}{\sqrt{P_\mu P^\mu + L^2}}\,. \tag{23}$$

After performing an integration by parts and substituting the dependent variables into Eq. (1) one obtains

$$S = \int d\tau \left[-G_{\mu\nu}X^\mu \dot{P}^\nu - \frac{\lambda^3}{2}G_{\mu\nu}X^\mu X^\nu\right]. \tag{24}$$

To quantize this system with the canonical formalism, the Dirac brackets [14, 15] must be constructed. In this process the Dirac brackets are replaced by commutators. Now, let us consider

$$\chi_1 = P_{1'} - L\,, \tag{25}$$
$$\chi_2 = X_{1'}\,, \tag{26}$$
$$\chi_3 = P \cdot X\,, \tag{27}$$
$$\chi_4 = \frac{1}{2}P^2\,, \tag{28}$$
$$\phi = \frac{1}{2}X^2\,. \tag{29}$$

A straightforward calculation shows that Eq. (29) is a first-class constraint while the others are second class. For the later ones we find

$$C_{\alpha\beta} \approx \{\chi_\alpha, \chi_\beta\} \approx \begin{pmatrix} 0 & -1 & -L & 0 \\ 1 & 0 & 0 & L \\ L & 0 & 0 & 0 \\ 0 & -L & 0 & 0 \end{pmatrix}, \tag{30}$$

from which,

$$C^{\alpha\beta} \approx -\frac{1}{L}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{L} \\ 0 & -1 & \frac{1}{L} & 0 \end{pmatrix}. \tag{31}$$

In general, given two functions A and B in phase space, the Dirac brackets are defined as

$$\{A, B\}^* = \{A, B\} - \{A, \chi_\alpha\}C^{\alpha\beta}\{\chi_\beta, B\}\,. \tag{32}$$

In particular, for the phase space coordinates

$$\{X_\mu\,, X_\nu\}^* = \frac{1}{L^2}\left(X_\mu P_\nu - X_\nu P_\mu\right), \tag{33}$$
$$\{X_\mu\,, P_\nu\}^* = \eta_{\mu\nu} + \frac{1}{L^2}P_\mu P_\nu\,, \tag{34}$$
$$\{P_\mu\,, P_\nu\}^* = 0\,. \tag{35}$$

These Dirac brackets are the classic version of the commutation rules of Snyder space [9]. Therefore, after quantizing the system we have the Snyder space as the background.

Now, by defining $\mathbf{X}_\mu = G_{\mu\nu}X^\nu$, the pair $(\mathbf{X}_\mu, P_\mu)$ satisfies the Dirac brackets

$$\{\mathbf{X}_\mu, \mathbf{X}_\nu\}^* = 0\,, \quad \{\mathbf{X}_\mu, P_\nu\}^* = \eta_{\mu\nu}\,, \quad \{P_\mu, P_\nu\}^* = 0\,. \tag{36}$$

That is, with the variables $(\mathbf{X}_\mu, P_\mu)$, the usual Poisson brackets are obtained. Nevertheless, at the quantum level the definition of $\mathbf{X}_\mu$ is ambiguous.

IV. EUCLIDEAN SNYDER SPACE

Other gauge conditions from which a noncommutative space can be obtained are

$$\chi_1 = P_0 - 1 = 0\,, \qquad \chi_2 = X_0 = 0\,. \tag{37}$$

For these the independent equations of motion are

$$\dot{X}^i = 0\,, \tag{38}$$
$$\dot{P}^i = -\lambda^3 X^i\,, \tag{39}$$
$$\phi_3 = g_{ij}X^i X^j \approx 0\,, \quad g_{ij} = \left(\delta_{ij} + \frac{P_i P_j}{1 - P_k P^k}\right) \tag{40}$$

and, as can be easily seen, the second-class constraints are given by

$$\chi_1 = P_0 - 1\,, \tag{41}$$
$$\chi_2 = X_0\,, \tag{42}$$
$$\chi_3 = P \cdot X\,, \tag{43}$$
$$\chi_4 = \frac{1}{2}P^2\,. \tag{44}$$

From a straightforward calculation it can be observed that in this case the matrix $C_{\alpha\beta} \approx \{\chi_\alpha, \chi_\beta\}$ is minus the matrix in Eq. (30) with $L = 1$. Using this, we find for the phase space coordinates

$$\{X_i\,, X_j\}^* = -\left(X_i P_j - X_j P_i\right), \tag{45}$$
$$\{X_i\,, P_j\}^* = \delta_{ij} - P_i P_j\,, \tag{46}$$
$$\{P_i\,, P_j\}^* = 0\,. \tag{47}$$

Thus, after quantizing the reduced system, a noncommutative space in the coordinates is obtained.

By defining the variable $\mathbf{X}_i = g_{ij}X^j$, it can be seen that the pair $(\mathbf{X}_i, P_j)$ satisfies

$$\{\mathbf{X}_i, \mathbf{X}_j\}^* = 0\,, \quad \{\mathbf{X}_i, P_j\}^* = \delta_{ij}\,, \quad \{P_i, P_j\}^* = 0\,. \tag{48}$$

Now, as the only gauge transformations permitted are of the type of Eq. (12), there is no gauge transformation which takes the Snyder space to this system. In this sense they are different physical systems.

V. $O(d+1)$ NON-LINEAR SIGMA MODEL IN ONE DIMENSION

It can be seen that the Hamiltonian H_{2T} from Eq. (2) is invariant under the transformations

$$(X^M, P^M) \to (P^M, X^M)\,, \quad (\lambda^1, \lambda^2, \lambda^3) \to (\lambda^3, \lambda^2, \lambda^1)\,. \tag{49}$$

This symmetry implies that if we impose gauge conditions and then the Xs and Ps are swapped, we obtain analogous reduced systems. Notice, however, that the physical interpretation of each one is different. As an example, by performing this swap in the gauge conditions of the Euclidean Snyder space from Eq. (37), one obtains

$$\chi_1 = X_0 - 1 = 0 \qquad \text{and} \qquad \chi_2 = P_0 = 0\,. \tag{50}$$

In this case the independent reduced equations of motion are

$$\dot{X}^i = \lambda^1 P^i\,, \qquad (i = 1, \ldots, d) \tag{51}$$
$$\dot{P}^i = 0\,, \tag{52}$$
$$\phi_1 = \tilde{g}_{ij}P^iP^j \approx 0, \quad \tilde{g}_{ij} = \left(\delta_{ij} + \frac{X_iX_j}{1 - X_kX^k}\right), \tag{53}$$

with dependent variables given by

$$X^{0'} = \sqrt{X^iX_i - 1}\,, \quad P^{0'} = \frac{(P^iX_i)}{\sqrt{X^iX_i - 1}}\,. \tag{54}$$

Now, by rewriting Eq. (1) in terms of the independent variables we obtain

$$S = \int_{\tau_1}^{\tau_2} d\tau \left[\tilde{g}_{ij}\dot{X}^iP^j - \frac{\lambda^1}{2}\tilde{g}_{ij}P^iP^j\right]. \tag{55}$$

From this expression one gets to the equations of motion (51)–(53). Now, substituting Eq. (51) into Eq. (55) and eliminating P_i as a dynamic variable we get

$$S = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau \frac{\tilde{g}_{ij}\dot{X}^i\dot{X}^j}{\lambda^1}\,. \tag{56}$$

Eq. (56) could be interpreted as the action of a massless free particle in a space with metric $\tilde{g}_{ij}$, but non-relativistic massless particles are not natural. In a better interpretation, for $\lambda^1 = 1$, this equation represents the action of the $O(d+1)$ non-linear sigma model in one dimension [17].

The Dirac brackets for the phase space coordinates, in this case, are

$$\{X_i\,, X_j\}^* = 0\,, \tag{57}$$
$$\{X_i\,, P_j\}^* = \delta_{ij} - X_iX_j\,, \tag{58}$$
$$\{P_i\,, P_j\}^* = -\left(X_iP_j - X_jP_i\right). \tag{59}$$

However, using the coordinates $(X_i, \bar{\mathbf{P}}_i = \tilde{g}_{ij}P^j)$ one gets to

$$\{X_i\,, X_j\}^* = 0\,, \quad \{X^i\,, \bar{\mathbf{P}}_j\}^* = \delta^i_j\,, \quad \{\bar{\mathbf{P}}_j\,, \bar{\mathbf{P}}_j\}^* = 0\,. \tag{60}$$

In terms of the variables $(X_i, \bar{\mathbf{P}}_i)$, the action of Eq. (55) becomes

$$S = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{X}^i\mathbf{P}_i - \frac{\lambda^1}{2}\tilde{g}^{ij}\mathbf{P}_i\mathbf{P}_j\right]. \tag{61}$$

Thus, this system can be thought of: either a particle in an Euclidean metric with a deformed Poisson structure, Eqs. (57)–(59), or as a particle in the metric $\tilde{g}_{ij}$ with the standard Poisson structure, Eq. (60). A similar interpretation can be given to the systems presented in sections 3 and 4.

By performing the change of variables $(X, P) \to (P, X)$ in the gauge conditions for the Snyder space, Eq. (18), one gets to the gauge conditions

$$X_{1'} = L = \text{const.}\,, \qquad P_{1'} = 0\,. \tag{62}$$

In Refs. [1] and [5] it is shown that, using the conditions from Eq. (62), the massless particle in the AdS space is obtained. This can also be easily verified by repeating the calculation using the conditions of Eq. (50).

It is remarkable that in the 2T model both dynamics in noncommutative spaces have as dual a dynamics in a curved space-time.

VI. SUMMARY

In this work we study a mechanical system with two times and gauge freedom called the two-time physics. It is shown that considering a particular gauge one gets a mechanical system with Dirac brackets consistent with the commutation rules of the Snyder noncommutative space. Using other gauge conditions an Euclidean version of the Snyder space is obtained. By exploiting a symmetry of the Hamiltonian we show that these noncommutative systems have a dual system. For the Snyder space, the dual is a massless particle in the AdS space, while for the Euclidean Snyder the dual is the non-linear sigma model in one dimension.

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